Int. J. Heat Mass Transfer. Vol. 20, pp. 1273–1280. Pergamon Press 1977. Printed in Great Britain

CONCERNING THE EFFECTIVE TRANSVERSE CONDUCTIVITY OF A TWO-DIMENSIONAL TWO-PHASE MATERIAL

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(*Received* 4 *October* 1976 *and in revised form* 25 *January* 1977)

Abstract—A two-phase material in which the phase boundaries are cylindrical surfaces is considered (fibre-reinforced material). We consider such a material for which the precise details of phase geometry are not known, and investigate to what extent knowledge of the effective transverse thermal conductivity for one pair of phase conductivities determines the effective conductivity for a geometrically identical material with different phase conductivities. Also considered is whether or not one can hope to obtain narrow bounds on effective conductivity in terms of three-point correlation functions.

NOMENCLATURE

- C, geometry dependent constants in power series expansion of effective conductivity;
- d, diameters in composite microstructure;
- e, parameter determining C_3;
- i, fraction determining nearness of composite microstructure to a limiting geometry;
- k, thermal conductivity;
- v, volume fraction.

Greek symbols

- α, β, structure parameters of model I;
- δ, $(k_1 - k_2)/k_2$;
- η, λ, structure parameters of model II.

Subscripts

- a, core;
- b, shell;
- n, term in power series expansion of effective conductivity;
- 1, 2, the two materials;
- 1, 2, 3, ..., terms in power series expansion of effective conductivity.

Superscripts

- *, effective value;
- ′ or ″, conductivity values for which aggregate effective conductivity is not known.

1. INTRODUCTION

THE PROBLEM of predicting the effective thermal conductivity of two-phase materials has engaged investigators for many years. Generally one deals with a composite material the phase geometry of which is not known exactly, it being random in nature. Recently techniques have been developed for introducing statistical geometric information in order to find solutions, or bounds, on thermal conductivity [1, 2]. However the statistical information required in these methods is far harder to acquire than simply to measure effective conductivity. A great advance was made earlier by Hashin and Shtrikman [3] for the three-dimensional case and by Hashin [4, 5] for the case to be considered here, i.e. the case of transverse conductivity of fibre-reinforced materials of arbitrary transverse phase geometry which are statistically isotropic in the transverse plane. They found bounds on thermal conductivity when only phase conductivities and volume fractions are known. They showed that their bounds were the best possible, i.e. the narrowest achievable, in the absence of any further knowledge about phase geometry. This they did by actually constructing models, mathematically, for which they could calculate precisely the effective conductivity and which realized their bounds. These models they called "composite spheres assemblages" and "composite cylinders assemblages" (CCA) respectively. Using a generalized version of their models the problem of predicting the effective transverse thermal conductivity of fibre-reinforced materials of arbitrary phase geometry will be examined from an entirely new point of view. Some interesting conclusions about the dependence of the transverse conductivity of two-phase fibre-reinforced materials will be drawn. The models proposed may seem to constitute poor representations of actual phase geometry. Indeed they are bad representations. It is not intended that the formulae presented should be used for predicting effective conductivity of two-phase composites. Rather these models have been chosen purely because their effective conductivities can be precisely calculated and because they can be constructed to produce any conductivity between the

Hashin bounds. In other words, instead of computing, approximately, effective conductivity for a given or assumed phase geometry the starting point is to find a family of phase geometries corresponding to any given effective conductivity between the Hashin bounds. This then provides the opportunity for shedding some light on the following question: if effective transverse conductivity is known for a particular composite for one set of phase conductivities, what can be said about the effective conductivity of a geometrically identical composite with different phase conductivities? The question is not simply academic since by virtue of the complete mathematical analogy of the problem under consideration with those of computing effective electrical conductivity, effective electrical permittivity and effective magnetic permeability (and indeed axial shear modulus [5]), we ask whether a measurement of any one of these effective "diffusion" constants (constituent constants being known) will cast light on the others. Since thermal conductivities are generally more difficult and time consuming to measure than these other quantities the conclusions to be drawn may prove useful.

The techniques pioneered in [1] and [2] for introducing statistical information in the form of correlation functions into the bounds on effective conductivity were later developed for the instance considered here, a transversely isotropic two-dimensional material [6–8]. Bounds were found in terms of three-point correlation functions. Schulgasser [9] was able to cast these bounds into a form containing only a single parameter of geometry additional to volume fraction. Left open was the question of the extent to which the proposed bounds exhausted the potential ability of three-point correlation functions in producing narrow bounds. Hashin and Shtrikman [3] and Hashin [4] had succeeded in constructing models which realized their bounds and they could thus demonstrate that their bounds exhausted all the possibilities inherent in the geometrical information they utilized, viz. the volume fractions (actually one-point correlation functions). It will be shown below that the value of the one parameter dependent on three-point correlation functions which appears in the bounds of [9] can be calculated for the models to be proposed. This enables one to investigate the question of the extent to which these proposed bounds have exhausted the possibilities inherent in the knowledge of three-point correlation functions.

2. CONSTRUCTION OF THE MODELS

Consider a two-phase material in which the phase boundaries are cylindrical surfaces. No other restriction is made on the geometry of phase boundaries. Indeed it is not even required that one phase be identifiable as a matrix, the other being the inclusion (fibre). It is assumed that the conductivity properties of each phase are isotropic and homogeneous and that there is perfect contact between the phases. Hashin [4, 5] showed that the bounds on the transverse conductivity (perpendicular to phase boundaries) of such a composite are given by

$$k_1 + \frac{2v_2k_1(k_2-k_1)}{2k_1+v_1(k_2-k_1)} \leqslant k^* \leqslant k_2 + \frac{2v_1k_2(k_1-k_2)}{2k_2+v_2(k_1-k_2)}. \quad (2.1)$$

k_1 and k_2 are the conductivities of the two phases with the indices being assigned such that $k_2 > k_1$. The volume fractions of the phases are v_1 and v_2 ($v_1+v_2 = 1$). Hashin further shows how a specific two-phase statistically isotropic and homogeneous material (called a composite cylinders assemblage) can be constructed from materials of conductivities k_a and k_b with volume fractions v_a and v_b such that its effective conductivity is given by

$$k^*_{a,b} = k_b + \frac{2v_ak_b(k_a-k_b)}{2k_b+v_b(k_a-k_b)}. \quad (2.2)$$

For $k_b = k_2$ and $k_a = k_1$ the Hashin upper bound is realized. For $k_b = k_1$ and $k_a = k_2$ the lower bound is realized. (The procedure for constructing such a material is presented in Appendix 1.) Hence the Hashin bounds are the best possible when the only known geometrical information is volume fractions.

The above bounds are based on an extension of classical variational principles which permits the utilization of the knowledge that the composite is transversely isotropic. Earlier bounds based on simpler classical principles were cruder because they did not permit this utilization.

By a procedure described in Appendix 1 one can generalize Hashin's composite cylinders assemblage model so as to arrive at a family of models, the transverse conductivities of which are given by

$$k^* = k_b + \frac{2k_b(\alpha v_1+\beta v_2)(k_a-k_b)}{2k_b+(1-\alpha v_1-\beta v_2)(k_a-k_b)}$$

where

$$k_a = k_2 + \frac{2k_2\alpha v_1(k_1-k_2)}{2k_2(\alpha v_1+\beta v_2)+\beta v_2(k_1-k_2)}$$

$$k_b = k_2 + \frac{2k_2(1-\alpha)v_1(k_1-k_2)}{2k_2(1-\alpha v_1-\beta v_2)+(1-\beta)v_2(k_1-k_2)}. \quad (2.3)$$

Here α and β are a pair of parameters which can, independently, assume any values from 0 to 1, each pair representing a different constructable material model. This family will be referred to as Ia.

Using equations (2.3) one can pose the question: given k^* (between the Hashin bounds) what pairs α, β (i.e. what *geometries*) will produce this effective conductivity? As an example consider the case $v_1 = v_2 = 0.5$ with $k_2 = 5$, $k_1 = 1$. The maximum and minimum transverse conductivities for such a material are 2.5 and 2 (Hashin bounds). In Fig. 1 lines of constant thermal conductivity (isokays) are shown in the α–β plane. It is clear that for any given k^* between the permitted values, an infinite number of distinct geometries will produce this effective conductivity within this one class of fibre-reinforced materials. The model will be used as a "test" against which to measure

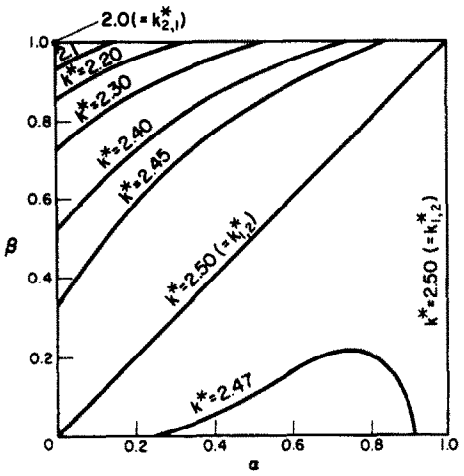

FIG. 1. Typical isokay diagram for Models Ia; $v_1 = v_2 = 0.5$, $k_2 = 5, k_1 = 1$.

several theoretical results. The general pattern of the isokays is similar for all ratios k_2/k_1. This is illustrated in Fig. 2 where the isokays for the ratios 1.1, 5, 15 and 100 are drawn for thermal conductivities which are located at definite fractions of the difference between the thermal conductivities of the extreme CCA models.

It is shown in Appendix 1 that another family of models can be constructed, the effective conductivities of which are given by

$$k^* = k_b + \frac{2k_b(1-\lambda v_1 - \eta v_2)(k_a - k_b)}{2k_b + (\lambda v_1 + \eta v_2)(k_a - k_b)}$$

where

$$k_a = k_2 + \frac{2k_2(1-\lambda)v_1(k_1-k_2)}{2k_2(1-\lambda v_1 - \eta v_2) + (1-\eta)v_2(k_1-k_2)}.$$
$$k_b = k_1 + \frac{2k_1\eta v_2(k_2-k_1)}{2k_1(\lambda v_1 + \eta v_2) + \lambda v_1(k_2-k_1)}. \quad (2.5)$$

Here again λ and η are a pair of parameters which can, independently, assume any values from 0 to 1, each pair representing a different constructable material model. This family of models will be referred to as IIa. A typical isokay diagram is shown in Fig. 3 for the same volume fractions and constituent conductivities as were employed in the example for the first models. Also shown is an isokay ($k^* = 4$) for the case $k_2/k_1 = 15$. It is noted that this isokay crosses the isokay $k^* = 2.3$ for the case $k_2/k_1 = 5$ at two points. This does not occur for Models I. If the indices 1 and 2 in equation (2.5) are interchanged another distinct model (IIb) is obtained.

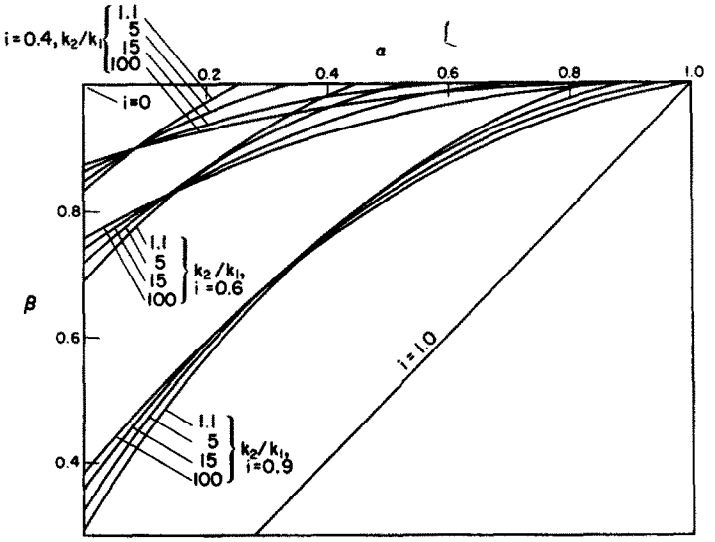

FIG. 2. Isokays for constant values of i for various ratios k_2/k_1. Models Ia, $v_1 = v_2 = 0.5$.

This fraction, which will again be referred to in Section 4, is defined by

$$i = \frac{k^* - k^*_{2,1}}{k^*_{1,2} - k^*_{2,1}}. \quad (2.4)$$

Other two-scale models can be constructed. One which is distinct from the above can be obtained by interchanging the indices 1 and 2 in equations (2.3). Isokay diagrams for such a material (IIb) are similar to those for the first described model except that high isokays appear where previously there were low ones and vice versa.

3. THE PRAGER PROBLEM

Prager [10] first posed and presented a solution to the problem of determining bounds on $k^{*\prime}$ for one set of phase conductivities when k^* is known for some other set of phase conductivities, nothing being known about phase geometry other than volume fractions. His results, obtained from classical variational principles, were derived for a statistically isotropic three-dimensional two-phase material but the development is equally valid line by line for the case under consideration here and the final results are unchanged.

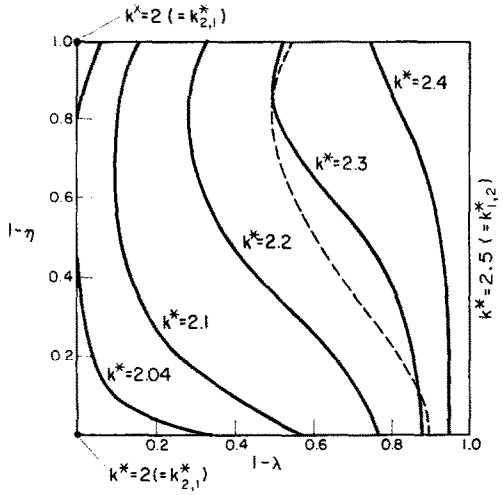

FIG. 3. Typical isokay diagram for Models IIa; $v_1 = v_2 = 0.5$, $k_2 = 5$, $k_1 = 1$. (Dashed line is isokay $k^* = 4$ for the case $k_2 = 15, k_1 = 1$.)

Prager's bounds [his equations (27) and (28)] are written out in full in Appendix 2 in our notation. These bounds are not narrow unless the ratio k_2/k_1 is similar in both the material for which k^* is known and for which it is sought. That narrower bounds can sometimes be found has been shown by Schulgasser [9]. It would be desirable if truly narrow bounds (or exact results) were achievable, i.e. that knowledge of k^* for one ratio k_2/k_1 would give meaningful information at far different ratios. That this unfortunately is not so can be conclusively shown by using the models presented in the previous section. Using a model described above, Model Ia for instance, all pairs α, β on a given isokay for a certain ratio k_2/k_1 can be used to find what $k^{*\prime}$s are realizable at the new ratio k_2'/k_1'. Typically, consider the example $v_1 = v_2 = 0.5$ with $k_2 = 15$ and $k_1 = 1$ and with k^* known to be 4. Then with k_2 changed to 50 the following results are obtained:

Hashin	$2.85 \leqslant k^* \leqslant 17.55$
Prager bounds	$5.32 \leqslant k^* \leqslant 10.37$
Realizable (Models Ia)	$5.98 \leqslant k^* \leqslant 9.49$

The Prager bounds are somewhat better than the Hashin bounds, but are not close enough to be meaningful. The use of Model Ia shows conclusively that it is not possible to improve appreciably on the Prager bounds. Maximum and minimum realizable values are found when $\alpha = 0$ and when $\beta = 1$ respectively. For the particular case $v_1 = v_2 = 0.5$, Models Ib always produce the same extreme bounds as those obtained from Ia. This is not true invariably and generally both must be investigated. It has been found that the extreme realizable values obtained from Models IIa and IIb are always within those attainable using Models Ia and Ib.

One can go even further than above and assume that k^* is known for two different ratios k_2/k_1. As an example we will once more take $v_1 = v_2 = 0.5$ with $k_2 = 15$ and $k_1 = 1$ and with $k^* = 4$; and further, when $k_2' = 5$ and $k_1' = 1$ take $k^{*\prime} = 2.3$. Considering Models IIa we refer to Fig. 3 and it is clear that two pairs λ, η (i.e. two different phase geometries) give these results. For each of these pairs one can examine $k^{*\prime\prime}$ for some other k_2'' and k_1''. For $k_2'' = 50$ and $k_1'' = 1$, the two values of $k^{*\prime\prime}$ obtained are 6.075 and 6.533. Thus, when k^* is known for two ratios k_2/k_1, a fairly precise determination of k^* at a third ratio may be possible. The Models Ia which produced the large spread when k^* is known at only one ratio k_2/k_1 does not admit double crossing of contour lines. Hence for this model if k^* is known at two ratios it is determinate at a third. Models Ia predict $k^{*\prime\prime}$ to be precisely 6.553. Models II do permit double crossing, but at the third ratio there is only a relatively small spread in realizable k^*'s. Since no claim is made that the models exhaust all possibilities, this is no guarantee that the maximum possible spread at the third ratio is indeed small. However since they are sufficiently flexible to have realized most of the Prager bounds (and, as has been pointed out [9] the Prager bounds are not necessarily the best obtainable) one would suspect that a calculation of the type performed above gives a fairly good indication of the possible spread in k^* at the third ratio.

4. BOUNDS BASED ON THREE-POINT CORRELATION FUNCTIONS

Beran and Silnutzer [6, 7] and, independently, Hori and Yonezawa [8] have found the same bounds on k^* in terms of integrals of three-point correlation functions of phase properties. The upper and lower bounds are found in terms of different three-point correlation functions. It was shown by Schulgasser [9] that both of these bounds can be written in terms of a single three-point correlation function. It was further shown that if k^*/k_2 is expanded in a power series in $\delta = (k_2 - k_1)/k_2$, i.e.

$$\frac{k^*}{k_2} = \sum_{n=1}^{\infty} C_n \delta^n, \tag{4.1}$$

the C_n being dependent only on phase geometry and not on phase conductivity, then the single correlation function can be related to C_3. The Hashin bounds, equation (2.1), expanded in a power series gives:

lower bound:

$$\begin{aligned} k^*/k_2 = 1 &- v_1\delta - (v_1v_2/2)\delta^2 \\ &- (v_1v_2/4)(1+v_2)\delta^3 - \ldots \\ &- (v_1v_2/2)\{(1+v_2)/2\}^{n-2}\delta^n - \ldots \end{aligned} \tag{4.2}$$

upper bound:

$$\begin{aligned} k^*/k_2 = 1 - v_1\delta - (v_1v_2/2)\delta^2 - (v_1v_2/4)\delta^3 - \ldots \\ - (v_1v_2/2)\{v_2/2\}^{n-2}\delta^n - \ldots . \end{aligned} \tag{4.3}$$

We see that the coefficients of the terms coincide up to the second order term and noting the form of the third order coefficient in the bounds we see that it is convenient to write C_3 in the form

$$C_3 = -\frac{v_1v_2}{4}(v_2 + 1 - e). \tag{4.4}$$

e can vary from 0 to 1. Then the bounds in [9] take the form

$$k_1k_2\left\{k_1v_2+k_2v_1-\frac{v_1v_2(k_2-k_1)^2}{2k_1+(v_2+1-e)(k_2-k_1)}\right\}^{-1} \leqslant k^* \leqslant k_1v_1+k_2v_2-\frac{v_1v_2(k_1-k_2)^2}{2k_2+(v_2+1-e)(k_1-k_2)}. \tag{4.5}$$

These are rigorous bounds which include a single parameter, e, of phase geometry additional to the very limited information about phase geometry used to obtain the Hashin bounds, i.e. volume fractions only. One would hope that knowledge of such additional geometric information would result in narrow bounds on k^*. This unfortunately is not the case. The bounds (4.5) are quite wide for $k_2/k_1 \gg 1$, and we will now show, by use of the models of Section 2, that while these may not be the very best bounds achievable in terms of e, they cannot be very far from the best achievable.

It is shown in [9] that

$$C_3 = -\frac{v_1v_2}{4}\left\{1+v_2-\lim_{k_2\to k_1}\frac{k^*-k^*_{2,1}}{k^*_{1,2}-k^*_{2,1}}\right\} \tag{4.6}$$

or, using equation (4.4)

$$e = \lim_{k_2\to k_1}\frac{k^*-k^*_{2,1}}{k^*_{1,2}-k^*_{2,1}}. \tag{4.7}$$

Comparing equation (4.7) with (2.4) we note that

$$e = \lim_{k_2\to k_1} i.$$

In other words an isokay diagram such as Fig. 2 for $k_2/k_1 \sim 1$ is a map of constant values of e, a factor dependent purely on phase geometry and upon which a narrower set of bounds than those of Hashin can be based. We can test whether bounds in terms of e much better than those of equation (4.5) are possible by comparing these bounds for each value e with realizable values of k^* for some $k_2/k_1 \gg 1$ for each such value of i. Such a comparison is made in Fig. 4 for the case $v_1 = v_2 = 0.5$ with $k_2 = 15$, $k_1 = 1$, using k^*'s realizable from Models Ia. It is clear that while the bounds may not be the best possible they certainly cannot be much improved since about two-thirds of the spread between them is realizable.

5. A SIMPLE ESTIMATE OF $k^{*\prime}$

It has been abundantly illustrated above that knowledge of k^* at one ratio k_2/k_1 gives no precise information on the value of $k^{*\prime}$ at some other ratio. However an examination of Fig. 2 indicates how a reasonable estimate of $k^{*\prime}$ can be made. It is seen that the i-contour lines for various ratios k_2/k_1 are similar. The same conclusion can be drawn for Models II. If one simply assumes that the contour lines are the same, one arrives at the following simple formula for $k^{*\prime}$:

$$\frac{k^{*\prime}-k^{*\prime}_{2,1}}{k^{*\prime}_{1,2}-k^{*\prime}_{2,1}} = \frac{k^*-k^*_{2,1}}{k^*_{1,2}-k^*_{2,1}}. \tag{5.1}$$

Estimates using this formula lie between the realizable effective conductivities using the proposed models,

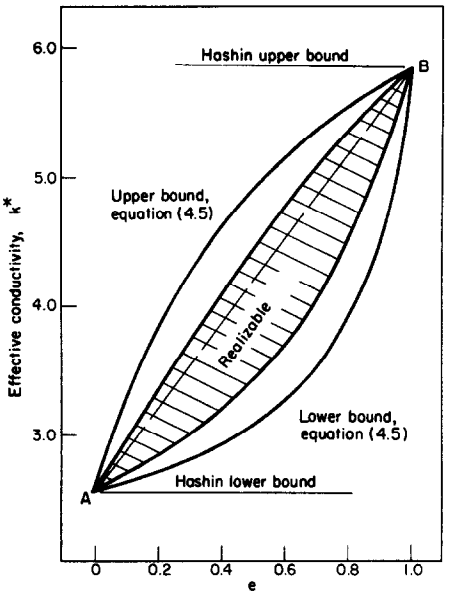

FIG. 4. Bounds and realizable values of the effective conductivity k^* as a function of the geometrical factor e; $v_1 = v_2 = 0.5$, $k_2 = 15$, $k_1 = 1$.

usually somewhere near the middle of the range. The implications of this estimate with respect to the bounds (4.5) are easy to interpret. Referring to Fig. 4 it is clear that (5.1) estimates $k^{*\prime}$ to lie on the straight line AB, when indeed the bounds range on both sides of this line. As k_2/k_1 approaches 1 the bounds can be shown to close in, approaching line AB [9]. It would then be expected that this estimate would be quite precise when both k_2/k_1 and k'_2/k'_1 are close to 1 or at least when $k_2/k_1 \sim k'_2/k'_1$. An example is given in Table 1 when neither of these conditions prevails. As is seen the estimate indeed falls in the middle of the possible range.

Table 1. Estimates and bounds on effective transverse conductivity for the case $v_1 = v_2 = 0.5$ and $k'_2 = 100$, $k'_1 = 1$ when it is known that for $k_2 = 50$, $k_1 = 1$, $k^* = 10$

	Lower bound	Estimate	Upper bound
Hashin	2.92		34.22
Prager	12.05		18.66
Realizable (Mods. Ia)	13.22		18.27
Equation (5.1)		16.08	

6. CONCLUSIONS

A set of models of two-phase two-dimensional composite materials has been devised which permits construction of materials corresponding to any possible effective thermal conductivity for given volume fractions and given constituent conductivities. By using these models it has been shown that:

(a) Knowledge of effective conductivity for one set of phase conductivities k_1, k_2 cannot afford precise knowledge or narrow bounds on effective conductivity for another set of phase conductivities.

(b) Knowledge of effective conductivity for two sets of phase conductivities probably affords fairly precise bounds on effective conductivity at a third set of phase conductivities.

(c) A reasonable estimate of effective conductivity for one set of phase conductivities can be made when effective conductivity is known at another set by use of equation (5.1).

(d) Knowledge of three-point correlation functions of two-phase composites cannot afford narrow bounds on effective conductivity of composite materials.

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APPENDIX 1

The cylinder assemblage model of Hashin is constructed as follows: Consider a space completely filled by composite cylinders, each cylinder having a core of conductivity k_a and an outer shell of conductivity k_b. The ratio of core diameter to outer diameter of the shell is maintained constant, i.e.

$$\frac{d_a}{d_b} = \text{constant},$$

and it is assumed that composite cylinders of all sizes are available so that in the limit the space can be completely filled out. This material is illustrated in Fig. 5. When the space is completely filled

$$v_a = \left(\frac{d_a}{d_b}\right)^3 \qquad v_b = 1 - \left(\frac{d_a}{d_b}\right)^3$$

and the effective conductivity of the aggregate is given by equation (2.2).

This model can be generalized as follows: Suppose that material having conductivities k_1 and k_2 are available in proportions determined by volume fractions v_1 and v_2. Let us take a portion of the k_1 material with volume fraction αv_1 ($\alpha \leqslant 1$) and a portion of the k_2 material with volume fraction βv_2 ($\beta \leqslant 1$) and construct a CCA material with the k_1

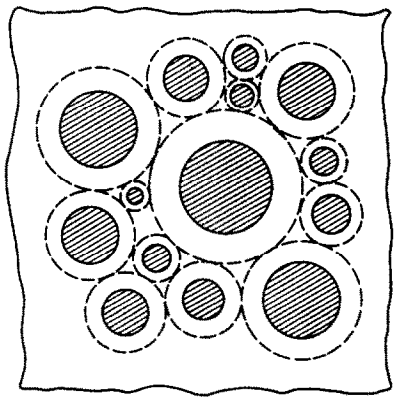

FIG. 5. Composite cylinders assemblage.

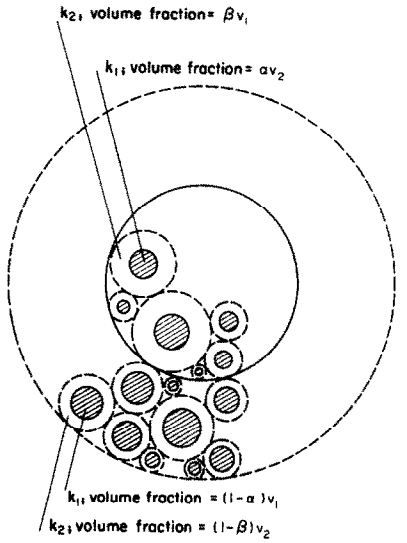

FIG. 6. Typical "larger scale" composite cylinder, Models Ia.

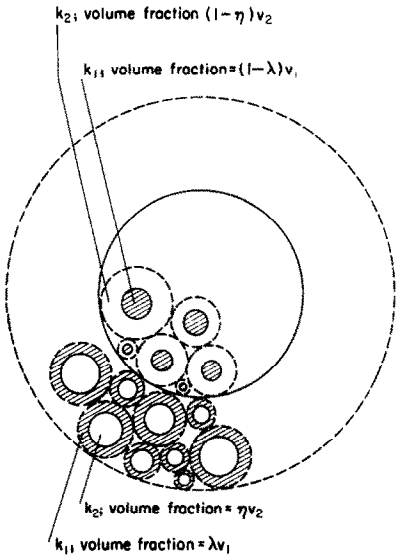

FIG. 7. Typical "larger scale" composite cylinder, Models IIa.

material being in the cores of the composite cylinders. The effective conductivity of such a material can be calculated using equation (2.2) with

$$v_a = \frac{\alpha v_1}{\alpha v_1 + \beta v_2} \qquad v_b = \frac{\beta v_2}{\alpha v_1 + \beta v_2}$$

and $k_a = k_1$, $k_b = k_2$. Now we take the remainder of the two materials and construct a second CCA material again with the k_1 material constituting the cores of the composite cylinders. The effective conductivity of such a material can again be calculated using equation (2.2) now taking

$$v_a = \frac{(1-\alpha)v_1}{1-\alpha v_1 - \beta v_2} \qquad v_b = \frac{(1-\beta)v_2}{1-\alpha v_1 - \beta v_2}$$

and again $k_a = k_1$, $k_b = k_2$. Now the basic assumption in seeking effective properties of composite materials is that as long as global geometry is large compared to microstructure the material can be treated as a continuum. Let us then take the two "continua" constructed above and construct from them another CCA material. It is only necessary that the scale of the diameters now being used be large compared to the scale of the two sub-composites. Into the core of this new CCA we put the first described sub-composite, into the shell the second. The appropriate volume fractions are

$$v_a = \alpha v_1 + \beta v_2 \qquad v_b = 1 - \alpha v_1 + \beta v_2.$$

A typical composite cylinder of this material is illustrated in Fig. 6 where the difference in scales has been understated. The effective conductivity of this two-scale composite is then given by (2.3). We here point out several interesting features of the model evident from Fig. 1. First of all for $\alpha = 1$, the effective conductivity is that of one version of the basic CCA model, $k^*_{1,2}$ (which is the Hashin upper bound when the phases are identified as above, or the lower bound had they been identified such that $k_2 < k_1$). For $\alpha = 0$, $\beta = 1$ the effective conductivity is that of the other version of the basic CCA model, $k^*_{2,1}$ (which is the Hashin lower bound when the phases are identified as above, or the upper bound had they been identified such that $k_2 < k_1$).

Another model is illustrated in Fig. 7. Here in the outer shell of the large scale composite cylinders the higher conductivity material is in the cores of the smaller scale composite cylinders while in the cores of the large scale composite cylinders the lower conductivity material is in the cores of the smaller scale cylinders. In the shells of the large scale composite cylinders, we put λv_1 of material "1" and ηv_2 of material "2". The effective conductivity of this two-scale composite is given by equation (2.5).

APPENDIX 2

For the case $k'_2/k'_1 > k_2/k_1$ Prager's bounds are

$$k^{*\prime} > \left[\left\langle \frac{1}{k'} \right\rangle - \frac{(v_2/k_1)(\langle 1/k\rangle - 1/k^*)(1/k'_1 - 1/k'_2)^2}{(v_2/k_1)(1/k_1 - 1/k_2)(1/k'_1 - 1/k'_2) + (1/k^* - 1/k_1)(1/k'_1 k_2 - 1/k'_2 k_1)}\right]^{-1}$$

$$k^{*\prime} < \langle k'\rangle - \frac{v_1 k_2(\langle k\rangle - k^*)(k'_1 - k'_2)^2}{v_1 k_2(k_1 - k_2)(k'_1 - k'_2) + (k^* - k_2)(k'_2 k_1 - k'_1 k_2)}$$

where $\langle\ \rangle$ denotes volume average, e.g. $\langle k\rangle = v_1 k_1 + v_2 k_2$ and $\langle 1/k\rangle = v_1/k_1 + v_2/k_2$. It is not required here that $k_2/k_1 > 1$. For the case $k'_2/k'_1 < k_2/k_1$ the proper bounds are obtained by interchanging the subscripts in the bounds written out above. Again there is no requirement that $k'_2/k'_1 > 1$.

A PROPOS DE LA CONDUCTIVITE EFFECTIVE TRANSVERSALE D'UN MATERIAU BIDIMENSIONNEL ET BIPHASIQUE

Résumé—On considère un matériau biphasique dans lequel les frontières entre phases sont des surfaces cylindriques (matériau renforcé par fibres). On étudie un matériau pour lequel les détails précis de géométrie de phase ne sont pas connus, et on cherche quel degré de connaissance de la conductivité thermique effective transversale, pour une paire de conductivités des phases, détermine la conductivité thermique effective d'un matériau identique avec différentes conductivités des phases. On regarde aussi si l'on peut espérer obtenir des relations étroites sur la conductivité effective en termes de fonctions de corrélation à trois points.

ZUR EFFEKTIVEN TRANSVERSALEN WÄRMELEITFÄHIGKEIT ZWEIDIMENSIONALER ZWEIPHASENMATERIALIEN

Zusammenfassung—Es wird ein Zweiphasenmaterial mit zylindrischen Phasengrenzflächen betrachtet (faserverstärktes Material), wobei die Phasengeometrie im Detail unbekannt ist. Es wird untersucht, inwieweit von der effektiven transversalen Wärmeleitfähigkeit für eine Materialkombination auf die effektive transversale Wärmeleitfähigkeit eines geometrisch identischen Materials mit anderer Wärmeleitfähigkeit der Komponenten geschlossen werden kann. Außerdem wird die Frage untersucht, ob man mit Hilfe von 3-Punkt-Korrelationsfunktionen enge Grenzen für die effektive Wärmeleitfähigkeit aufstellen kann.

ОБ ЭФФЕКТИВНОЙ ПОПЕРЕЧНОЙ ТЕПЛОПРОВОДНОСТИ ДВУХФАЗНОГО МАТЕРИАЛА

Аннотация — Рассматривается двухфазный материал, в котором фазовыми границами являются цилиндрические поверхности (материал, армированный волокнами). Выбран такой материал, для которого точно неизвестна геометрия фаз, и нами исследуется, в какой степени знание эффективной поперечной теплопроводности одной пары фаз позволяет определить эффективную теплопроводность геометрически идентичного материала с отличной теплопроводностью фаз. Рассматривается также возможность определения пределов эффективной теплопроводности с помощью трехточечных корреляционных функций.